MODEL OF A DISTENSIBLE TUBE ADMITTING LOCALIZED WAVES

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Solitary waves (in particular, solitons) play in a well-known sense in the analysis of substantially nonlinear systems the same fundamental role as normal modes play in the linear case. The corresponding methods (exact and approximate) are now well developed [1-3]. In this paper we propose a model for a distensible tube (pipeline), which takes into account the formation of such waves during the transportation of liquid. The possibility of the formation of waves of qualitatively different types is studied. Numerical experiments revealed their "soliton properties" and the process of localization is followed for some typical initial conditions.

Interest in such systems stems both from the possible technical applications and the definite relationship with problems in biomechanics: in blood vessels localized waves can be felt quite far from the source (pulse), and this apparently indicates their nonlinear nature.

1. Basic Assumptions. We shall study the axisymmetric flow of an ideal incompressible liquid, confined in a tube, whose surface at each moment in time t is a surface of revolution $r = a(x, t)$ relative to the x axis (Fig. 1). In the undeformed state the tube is assumed to have a radius $r = a^0 = \text{const}$, while the deformed (under the action of the pressure of the liquid) surface is such that the following condition of smoothness holds (the subscript here and below indicates differentiation with respect to the variable indicated):

$$|a_x| \ll 1. \tag{1.1}$$

Let us assume that the tube surface is formed by elastic rings, mounted on identically stretched strings (Fig. 1). We shall assume that the surface is continuous and impenetrable to the liquid. This construction can serve as a model of a cylindrical wafer shell or pipe made of an isotropic material, placed in an elastic medium.

We note that in the particular case when there are no strings for small displacements $(a - a^0)$ we obtain the model of an extensible pipe studied in [4]. The corresponding limiting process is analyzed below.

2. Derivation of the Equations of Dynamics of the Tube. According to the assumptions adopted in Sec. 1, we shall write the expression for the potential energy of the axisymmetric deformations in the form

$$\int_l\int_{S^*} \frac{T^*}{2} a_x^2 dS^* dx + \int_l\int_S \frac{\lambda\varepsilon^2}{2} dS dx = \pi\int_l aT^* a_x^2 dx + \pi\lambda a^0 \int_l \varepsilon^2 dx, \quad \varepsilon = \frac{dS^* - dS}{dS} \tag{2.1}$$

where $dS^* = a d\theta$, $dS = a^0 d\theta$ are elements of the length of the circumference in the transverse cross section in the deformed and starting states of the tube, respectively; T^* is the total force in the strings per unit length of the deformed ring; λ is the elastic constant, characterizing the (in this case linearly elastic) material of the rings; and, l is the length of the tube.

In view of the assumption that the liquid is ideal, the liquid exerts on the interior surface only a normal pressure $p^a = p^a(x, t)$ (we assume that the external pressure equals zero). Because of the mild slope we shall neglect the effect of this pressure and the forces in the rings on the change in the forces in the axial direction x, and we thus obtain

$$2\pi a T^* = 2\pi a^0 T = \text{const} \tag{2.2}$$

which is the condition for conservation of the flux of axial forces.

Taking into account (2.2) the expression (2.1) for the potential energy assumes the form

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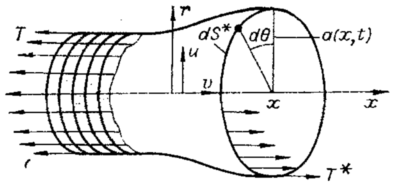

Fig. 1

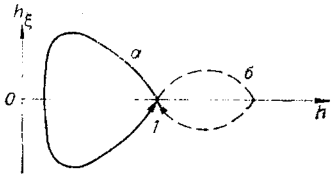

Fig. 2

$$\pi a^0 T \int_l a_x^2 dx + \pi a^0 \lambda \int_l \left(\frac{a}{a^0} - 1\right)^2 dx. \tag{2.3}$$

Let σ be the surface density of the pipe (σ^* is the same quantity in the deformed state). The condition for conservation of mass leads to the equality $2\pi a \sigma^* dx = 2\pi a^0 \sigma dx$, and the expression for the kinetic energy of the entire tube becomes

$$\int_l \int_{S^*} \frac{1}{2} a_t^2 \sigma^* dS^* dx = \pi a^0 \sigma \int_l a_t^2 dx. \tag{2.4}$$

The relations (2.3) and (2.4) permit writing the Lagrangian of the free system in the form

$$L = \frac{\sigma}{2} a_t^2 - \frac{T}{2} a_x^2 - \frac{\lambda}{2} \left(\frac{a}{a^0} - 1\right)^2. \tag{2.5}$$

The equation of motion is

$$\frac{d}{dt} \frac{\partial L}{\partial a_t} + \frac{\partial}{\partial x} \frac{\partial L}{\partial a_x} - \frac{\partial L}{\partial a} = Q^a. \tag{2.6}$$

Here $Q^a = p^a a / a^0$ is the generalized pressure, determined from the expression from the work performed by the pressure on the virtual displacement δa:

$$\int_l p^a \delta a 2\pi a dx = 2\pi a^0 \int_l p^a \frac{a}{a^0} \delta a\, dx = 2\pi a^0 \int_l Q^a \delta a\, dx.$$

Substituting (2.5) into (2.6) we obtain the final equation of motion

$$p^a = \frac{a^0}{a} \left[\sigma a_{tt} - T a_{xx} + \frac{\lambda}{a^0} \left(\frac{a}{a^0} - 1\right)\right]. \tag{2.7}$$

If $y = a - a^0$ is much smaller than a^0, then, linearizing (2.7) with respect to y, we arrive at the equation for oscillations of the string on a linearly elastic foundation

$$p^a = \sigma y_{tt} - T y_{xx} + \frac{\lambda}{a^{02}} y. \tag{2.8}$$

If, in addition, there are no strings (T = 0), Eq. (2.8) assumes the form of the equation of motion of a linear oscillator. This simplified model of the tube is used in [4].

3. Equations for the Flow and Formulation of the Problem. Euler's equations, averaged under the assumptions (1.1), for an axisymmetric flow in a cylindrical coordinate system have the form

$$a a_t + \frac{1}{2} (a^2 \langle v \rangle)_x = 0; \tag{3.1}$$

$$\langle v \rangle_t + \langle v \rangle \langle v \rangle_x = -\frac{1}{\rho} p^a_x \tag{3.2}$$

where $\langle v \rangle = \frac{2}{a^2} \int_0^a r v dr$ is the axial velocity, averaged over the cross section of the flow, of the particles; ρ is the density of the liquid. The Eqs. (2.7), (3.1), and (3.2) form a system relative to the quantities p^a, $\langle v \rangle$, and a. The pressure function is easily eliminated by substituting (2.7) into (3.2).

We introduce the dimensionless quantities

$$h = \frac{a}{a^0},\quad V = \frac{\langle v\rangle}{v^0},\quad \bar{x} = \frac{x}{a^0},\quad \bar{t} = \frac{v^0 t}{a^0},$$
$$\bar{\lambda} = \frac{\lambda}{\rho a^0 v^{02}},\quad \bar{T} = \frac{T}{\rho a^0 v^{02}},\quad \mu = \frac{\sigma}{\rho a^0}$$

(v^0 is the velocity of the steady-state flow in a section with radius $a = a^0$). Then after a simple transformation the system assumes the form (we omit the overbar on x, t, λ, and T)

$$V_t + VV_x = -\left[\frac{1}{h}(\mu h_{tt} - Th_{xx} - \lambda)\right]_x,\qquad h_t + Vh_x = -\frac{h}{2}V_x. \tag{3.3}$$

The system (3.3) admits a stationary solution with values of V and h (V = 1 and h = 1) which are constant along the x axis.

Assume that initially (at time t = 0) the velocity of the flow and the surface of the tube are perturbed. The problem consists of studying the further evolution of the form and propagation of these disturbances.

4. Solitary Waves. Analysis of the system (3.3) in the case of an infinite tube shows the possibility of propagation of solitary waves. Figure 2 shows the trajectories of the corresponding solutions in the plane (h_ξ, h), where ξ indicates differentiation with respect to the phase variable of the traveling wave.

The type of trajectory depends on the value of the parameter $b = (1 - c)^2/2\lambda$. For $b < 1/4$ the trajectory (a) lies in the region $h < 1$. It corresponds to a solitary wave in the form an axisymmetric dent in the tube. If $1/4 < b < 1$, h is greater than unity (b). This is a traveling wave in the form of a localized axisymmetric "bulge." In Fig. 2 the loops a and b as $b \to 1/4$ converge to a point h = 1, while for b = 1/4 the possible form of the solitary wave changes qualitatively. For a values of b close to 1/4 a simple analytic solution can be obtained

$$h = 1 - \frac{A}{\mathrm{ch}^2\,\xi},\quad V = c + \frac{1-c}{h^2}, \tag{4.1}$$

where A = (1/4b) − 1

$$\xi = \frac{1}{2}\sqrt{\lambda\left(\frac{1-4b}{T-\mu c^2}\right)}\,(x - ct). \tag{4.2}$$

Analysis of this solution and the radicand in (4.2) shows that the localized dent propagates in quite rigid or relatively light tubes.

Conversely, for propagation of a localized bulge the tube must be quite massive or it must have a relatively high stiffness for transverse stretching (compared with axial stretching).

The following limitation is necessary in using the solution (4.1):

$$|A| = \left|\frac{\lambda}{2(1-c)^2} - 1\right| \ll 1.$$

In the absence of longitudinal stretching (T = 0) only the second type of solitary wave is possible.

5. Computational Scheme. Numerical experiments were performed in order to study the character of wave processes for initial disturbances of a more general form. The system (3.3) was integrated in the particular case of a zero-inertia tube (μ = 0) with periodic boundary conditions.

An implicit three-step finite difference integration scheme with second-order accuracy of the approximation of spatial derivatives was selected. The system of equations was integrated on a 101-point grid with a step $\Delta x = 0.01$ in x and $\Delta t = 0.001$ in time, which guaranteed that the computational scheme will be stable. The initial conditions were given in the form (4.1). In Figs. 3 and 4 the numbers in the upper left-hand corner of each frame indicate the number of the integration step in time. The values of the ordinates of points on the surface of the tube h = h(x) are indicated on the left (solid lines), and the numbers on the right correspond to the graphs of the velocity distribution V = V(x) (broken lines).

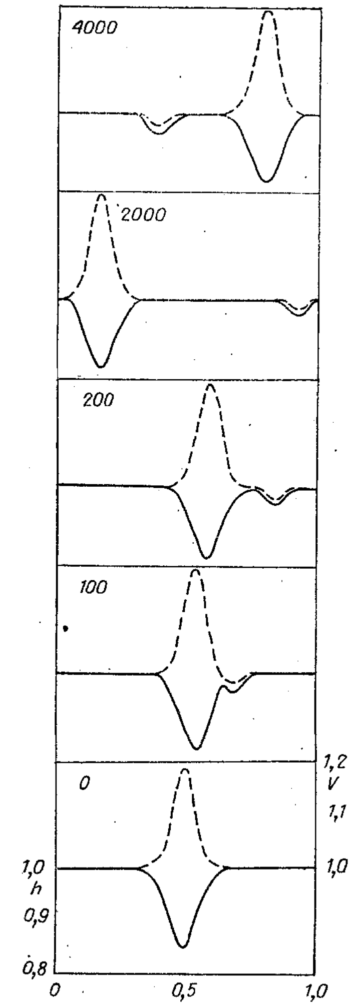

Fig. 3

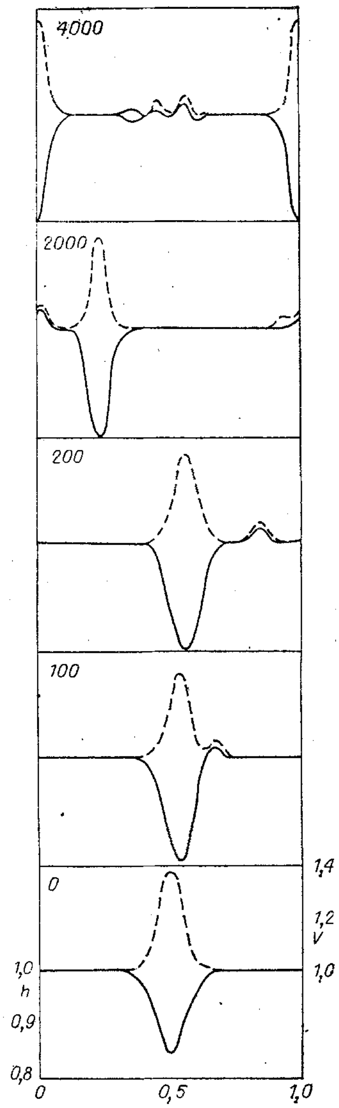

Fig. 4

6. Discussion. The mutual symmetry of the graphs of the velocity and of the surface relative to the abscissa axis indicates that at the locations of the constrictions the velocity is higher than in the bulges of the tube. However, "surges" were found in the form of localized waves, for which the convexities of the graphs are directed in the same direction – downwards or upwards (Figs. 3 and 4). In spite of the small dimensions compared with the dent, bulges propagate with a higher phase velocity. We emphasize that the calculations were performed for the case of a zero-inertia pipe ($\mu = 0$), when the analysis described in Sec. 4 gives solitary waves only in the form of dents.

In all cases there is a tendency for localized waves to form from different initial disturbances. These waves have a definite stability and retain their individuality for quite a long time after interaction with waves similar to them. The process of shedding of the "excess energy" by the initial disturbance is analogous to the process demonstrated by some exactly integrable systems [1, 2].

On the whole the results of the numerical analysis show that the behavior of localized waves in the system under study has the basic characteristic features of the propagation of solitons, at least over relatively long time intervals.

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SELF-SIMILAR SOLUTIONS TO THE PROBLEM OF THE MOTION OF A SPHERICAL PISTON IN A HEAT-CONDUCTING MEDIUM WITH CONSERVATION OF THE ENERGY OF A POINT EXPLOSION

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In this paper we study the problem of the motion of a spherical piston with fixed heat removal on it along a heat-conducting medium with a distributed density, in which there initially occurred a point explosion which released a finite energy E_0. We study the case when the heat removal is compensated by the work performed by the piston, i.e., the total energy of the medium remains constant and equal to the released energy E_0.

Analysis of the numerically found self-similar solutions revealed the following features.

For solutions which have the same total energy, as the velocity of the piston and the rate of heat removal on it increase, the mass velocity or propagation of the forward wave front, the difference between the velocity of the forward front of the perturbations and the velocity of the shock wave following it, and the relative fraction of the thermal energy all decrease.

As E_0 increases, first of all, the behavior indicated above intensifies and, second, interesting features are observed for two limiting problems – a pure explosion [1, 2] and maximum heat removal: the percentage of the kinetic energy of the explosion in the problem without the piston (pure explosion) drops and the percentage of the kinetic energy of the explosion in the problem with maximum heat removal (the temperature at the piston equals zero) increases.

We write the system of gas-dynamics equations in the Lagrangian mass coordinate system [3] in the form

$$\frac{\partial r}{\partial t} = v,\ \frac{\partial r}{\partial m} = \frac{1}{\rho r^2},\ \frac{\partial v}{\partial t} = -r^2\frac{\partial p}{\partial m},\ \frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = \frac{\partial (r^2 v)}{\partial m},$$
$$\frac{\partial \varepsilon}{\partial t} = -p\frac{\partial (r^2 v)}{\partial m} - \frac{\partial (r^2 w)}{\partial m},\ W = -r^2\rho\varkappa\frac{\partial T}{\partial m}, \tag{1}$$
$$\varepsilon = \frac{RT}{\gamma - 1},\ p = R\rho T,\ \varkappa = aT^{5/2},\ \gamma = \frac{5}{3}.$$

Here r is the radius, m is the Lagrangian mass variable, t is the time, v is the velocity, ρ is the density, p is the pressure, ε is the internal energy, T is the temperature, W is the heat flux, and $\varkappa$ is the coefficient of thermal conductivity, characteristic for a high-temperature hydrogen plasma.

Dimensional analysis [4] shows that the problem of an instantaneous point explosion followed by the motion of a spherical piston has a self-similar solution when the following hold:

boundary conditions on the piston ($m = 0$)

$$v(0,\ t) = v_0 t^{-1/4},\ W(0,\ t) = -v(0,\ t)p(0,\ t); \tag{2}$$

boundary conditions on the forward front of the perturbation wave $m_N(t)$

$$\rho(m_N,\ t) = \rho_0 m_N^{-7/2},\ v(m_N,\ t) = T(m_N,\ t) = W(m_N,\ t) = 0; \tag{3}$$

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